CEE 6040: Reliability Analysis & Prob. Modeling

6. Reliability analysis III: *Hasofer-Lindt algorithm*

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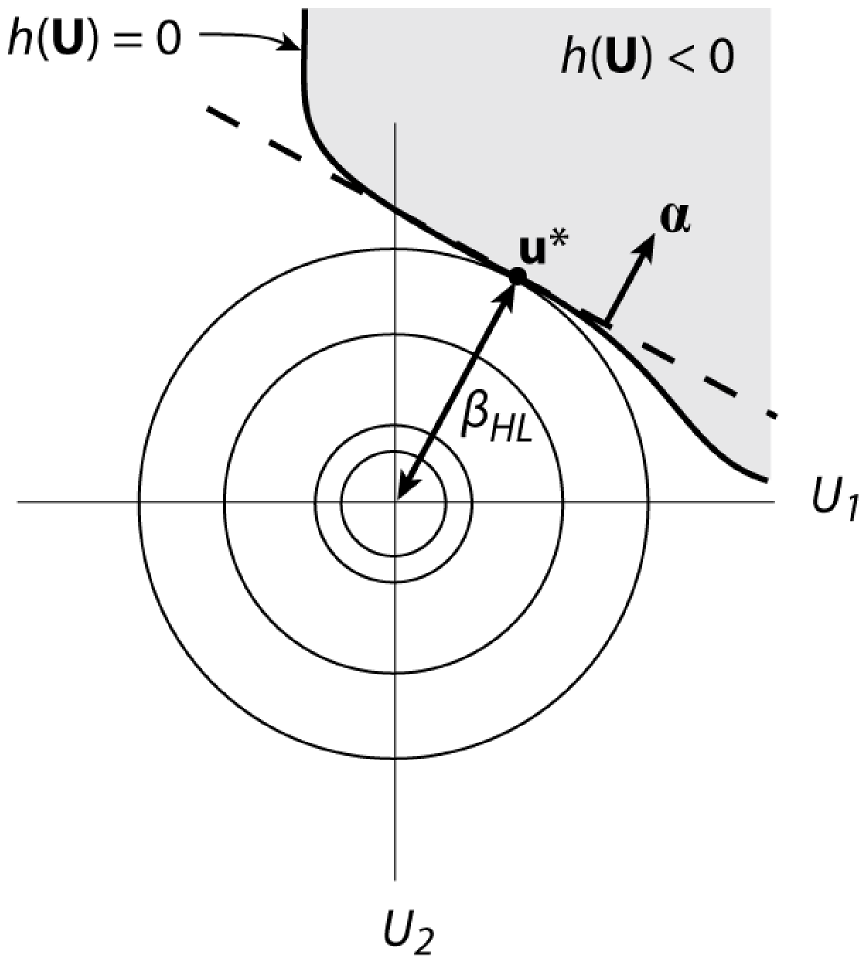
Motivation

Motivated by the geometric interpretation of the β_{MVFOSM} index for linear limit state functions, Hasofer and Lind (1974) suggested that we linearize nonlinear limit state functions at $\mathbf{u}^*$, the nearest point on the limit state surface $h(\mathbf{u})$.

- It is the most likely failure point (in a second-moment sense)
- As long as the limit state *surface* does not change, the answer will be invariant to the limit state *function*
- The only difference between this approach and the MVFOSM will be the **linearization point**



The Hasofer-Lindt Reliability Index

To find β_{HL}, we are going to need to find $\mathbf{u}^*$ (and its counterpart, $\mathbf{x}^*$) This is an optimization problem:

$$\text{minimize } \|\mathbf{u}\|$$
$$\text{subject to } h(\mathbf{u}) = 0$$

Once we find $\mathbf{u}^*$, we can find the corresponding $\mathbf{x}^*$. Then we can take a Taylor series expansion around $\mathbf{x}^*$

$$g(\mathbf{X}) \cong g(\mathbf{x}^*) + \sum_{i=1}^{n} (X_i - x_i^*) \left.\frac{\partial g}{\partial x_i}\right|_{\mathbf{x}=\mathbf{x}^*}$$
$$\cong \nabla g^T(\mathbf{x}^*)(\mathbf{X} - \mathbf{x}^*)$$

Thus:

$$\mu_{g(\mathbf{X})} \cong \nabla g^T(\mathbf{x}^*)(\mathbf{M} - \mathbf{x}^*)$$

$$\sigma_{g(\mathbf{X})}^2 \cong \left(\nabla g^T(\mathbf{x}^*)\right)\Sigma\left(\nabla g(\mathbf{x}^*)\right)$$

$$\beta_{HL} = \frac{\nabla g^T(\mathbf{x}^*)(\mathbf{M} - \mathbf{x}^*)}{\sqrt{\left(\nabla g^T(\mathbf{x}^*)\right)\Sigma\left(\nabla g(\mathbf{x}^*)\right)}}$$

Or we can just take the Taylor Series expansion in standard space:

$$h(\mathbf{U}) \cong h(\mathbf{u}^*) + \sum_{i=1}^{n} (U_i - u_i^*) \left.\frac{\partial h}{\partial u_i}\right|_{\mathbf{u}=\mathbf{u}^*}$$

$$\cong \nabla h^T(\mathbf{u}^*)(\mathbf{U} - \mathbf{u}^*)$$

Thus:

$$\mu_{h(\mathbf{U})} \cong -\nabla h^T(\mathbf{u}^*)(\mathbf{u}^*)$$

$$\sigma_{h(\mathbf{U})}^2 \cong \left(\nabla h^T(\mathbf{u}^*)\right)\left(\nabla h(\mathbf{u}^*)\right) = \|\nabla h(\mathbf{u}^*)\|^2$$

$$\beta_{HL} = \frac{-\nabla h^T(\mathbf{u}^*)(\mathbf{u}^*)}{\|\nabla h(\mathbf{u}^*)\|} = \boldsymbol{\alpha}^T \mathbf{u}^*$$

where:

$$\boldsymbol{\alpha}^T = \frac{-\nabla h^T(\mathbf{u}^*)}{\|\nabla h(\mathbf{u}^*)\|}$$

Finding u*: the Hasofer-Lind Rackwitz-Fiessler Algorithm

The Hasofer-Lind Rackwitz-Fiessler (HL-RF) Algorithm is one of several algorithms that can solve our required optimization problem

$$\begin{aligned}&\text{minimize } \|\mathbf{u}\| \\ &\text{subject to } h(\mathbf{u}) = 0\end{aligned}$$

HL-RF is a recursive algorithm that finds successively better **u** values. Starting from some initial guess **u**(0), we iterate to $\mathbf{u}^*$ using the recursive formula:

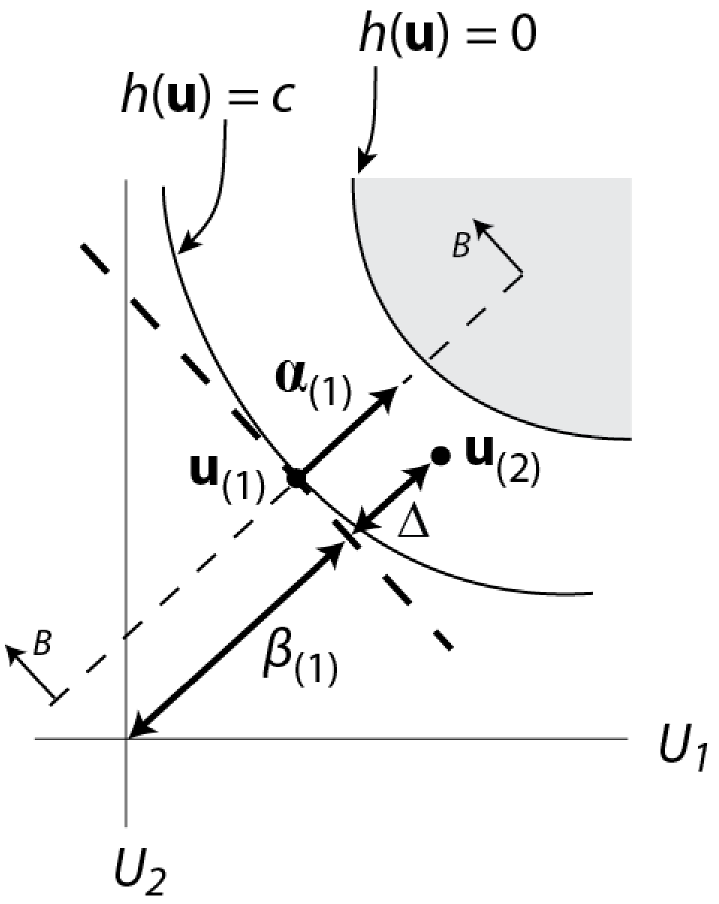
$$\mathbf{u}_{(k+1)} = \boldsymbol{\alpha}_{(k)}\left[\beta_{(k)} + \frac{h(\mathbf{u}_{(k)})}{\|\nabla h(\mathbf{u}_{(k)}\|}\right]$$

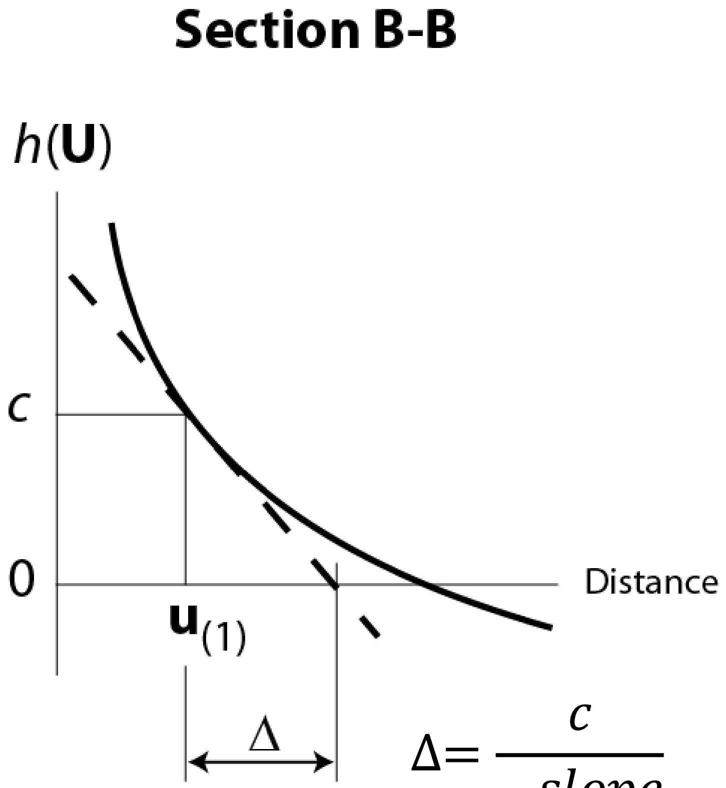
where:

$$\boldsymbol{\alpha}_{(k)} = \frac{-\nabla h(\mathbf{u}_{(k)})}{\|\nabla h(\mathbf{u}_{(k)})\|}$$

$$\beta_{(k)} = \boldsymbol{\alpha}_{(k)}^T \mathbf{u}_{(k)}$$

$$\nabla h(\mathbf{u}_{(k)}) = \begin{bmatrix} \partial h / \partial u_1 \\ \partial h / \partial u_2 \\ \vdots \\ \partial h / \partial u_n \end{bmatrix}_{\mathbf{u}=\mathbf{u}_{(k)}}$$





$$\Delta = \frac{c}{-slope}$$

$$\mathbf{u}_{(k+1)} = \boldsymbol{\alpha}_{(k)}\left[\beta_{(k)} + \frac{h(\mathbf{u}_{(k)})}{\|\nabla h(\mathbf{u}_{(k)})\|}\right]$$

$$\boldsymbol{\alpha}_{(k)} = \frac{-\nabla h(\mathbf{u}_{(k)})}{\|\nabla h(\mathbf{u}_{(k)})\|} \qquad \beta_{(k)} = \boldsymbol{\alpha}_{(k)}^{T}\mathbf{u}_{(k)}$$

Finding the gradient of $h(\mathbf{u})$

$$\nabla h(\mathbf{u}_{(k)}) = \begin{bmatrix} \partial h / \partial u_1 \\ \partial h / \partial u_2 \\ \vdots \\ \partial h / \partial u_n \end{bmatrix}_{\mathbf{u}=\mathbf{u}_{(k)}}$$

and

$$\frac{\partial h}{\partial u_1} = \frac{\partial g}{\partial x_1}\frac{\partial x_1}{\partial u_1} + \frac{\partial g}{\partial x_2}\frac{\partial x_2}{\partial u_1} + \cdots + \frac{\partial g}{\partial x_n}\frac{\partial x_n}{\partial u_1}$$

so

$$\nabla h(\mathbf{u}_{(k)}) = \mathbf{J}_{\mathbf{xu}}^T \nabla g(\mathbf{x}_{(k)})$$

$$= (\mathbf{DL})^T \nabla g(\mathbf{x}_{(k)})$$

where

$$\mathbf{J}_{\mathbf{xu}} = \begin{bmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_1}{\partial u_2} & \cdots & \frac{\partial x_1}{\partial u_n} \\ \vdots & \ddots & & \vdots \\ \frac{\partial x_n}{\partial u_1} & \cdots & & \frac{\partial x_n}{\partial u_n} \end{bmatrix}$$

✓ We stop iterating when the following conditions are satisfied:

$$|\beta_{(k+1)} - \beta_{(k)}| < \varepsilon_1$$

$$|h(\mathbf{u}_{(k)})| < \varepsilon_2$$

Summary: Computing β_{HL}

1) Identify limit state function, and means and covariances for random variables
2) Transform limit state function and random variables into standard Space
3) Find $\mathbf{u}^*$ (using, e.g., HL-RF algorithm)
4) Linearize h($\mathbf{u}$) at $\mathbf{u}^*$ and compute second-moment reliability index

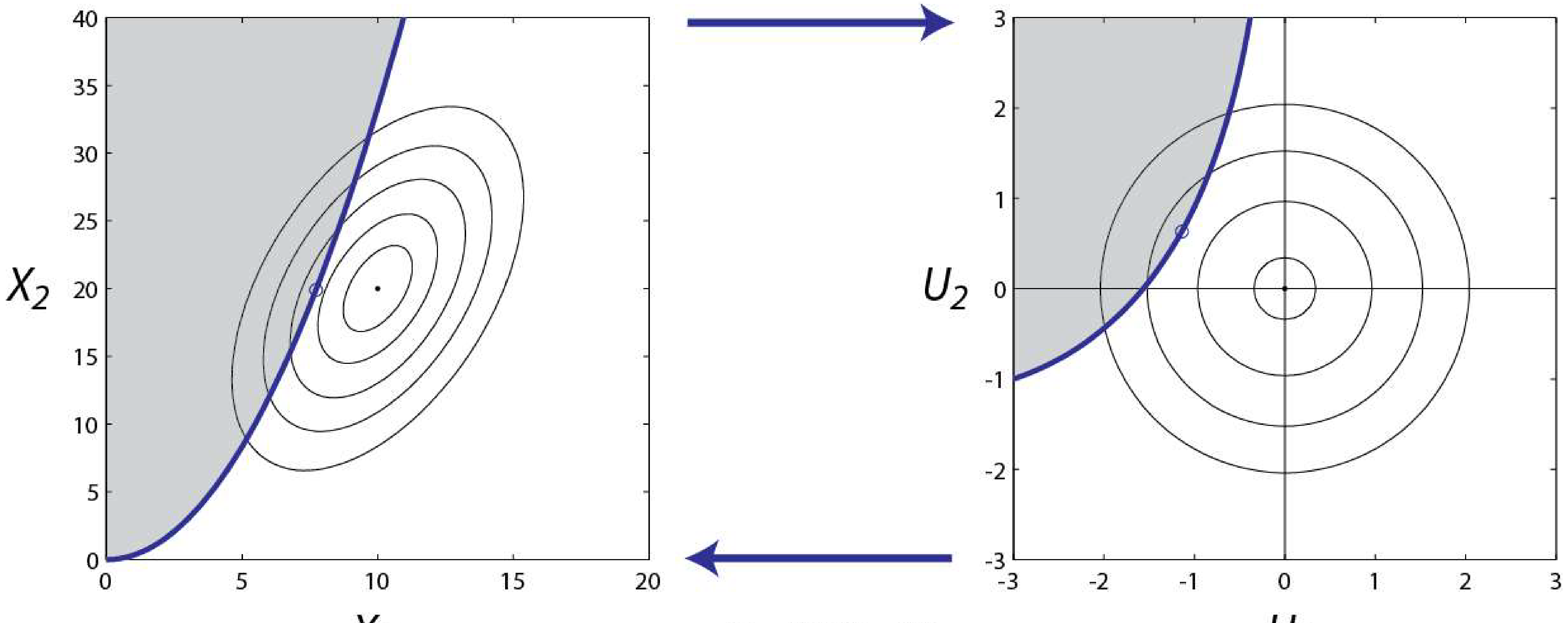
Example: Consider the limit state function $g(x_1, x_2) = {x_1}^2 - 3x_2$, where the random variables X_1 and X_2 have the following moments:

$$\mu_{X_1} = 10 \qquad \mu_{X_2} = 20$$

$$\sigma_{X_1} = 2 \qquad \sigma_{X_2} = 5 \qquad \rho_{X_1,X_2} = 0.5$$

Find the Hasofer-Lind reliability index

$$\mathbf{U} = \mathbf{L}^{-1}\mathbf{D}^{-1}(\mathbf{X} - \mathbf{M})$$



$$\mathbf{X} = \mathbf{DLU} + \mathbf{M}$$

Example: Consider the limit state function $g(x_1, x_2) = 1 - 3x_2/(x_1^2)$, where the random variables X_1 and X_2 have the following moments:

$$\mu_{X_1} = 10 \qquad \mu_{X_2} = 20$$

$$\sigma_{X_1} = 2 \qquad \sigma_{X_2} = 5 \qquad \rho_{X_1,X_2} = 0.5$$

Find the Hasofer-Lind reliability index

Comments on β_{HL}:

1. We solved the invariance problem
2. We have only incorporated second moment information, so we don't know the true probability of failure

Comparisons of our different reliability indices

(a) When $g(\mathbf{X})$ is linear $\beta_{MVFOSM} = \beta_{HL}$

(b) When $g(\mathbf{X})$ is linear and **X** has a multivariate normal distribution

$$\beta_{MVFOSM} = \beta_{HL} = \beta_{true}$$

(c) When $g(\mathbf{X})$ is nonlinear $\beta_{MVFOSM} \neq \beta_{HL} \neq \beta_{true}$

β_{MVFOSM} likely not invariant
β_{HL} might be OK